Journal of Statistical Physics, Vol. 127, No. 5, June 2007 (© 2007)
DOI: 10.1007/s10955-007-9315-z

Functional Feynman-Kac Equations for Limit Lognormal Multifractals

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Received November 1, 2006; accepted March 8, 2007
Published Online: April 3, 2007

A novel technique of functional Feynman-Kac equations is developed for the probability distribution of the limit lognormal multifractal process introduced by Mandelbrot [in *Statistical Models and Turbulence*, M. Rosenblatt and C. Van Atta, eds., Springer, New York (1972)] and constructed explicitly by Bacry, Delour, and Muzy [*Phys. Rev. E* **64**:026103 (2001)]. The distribution of the process is known to be determined by the complicated stochastic dependence structure of its increments (SDSI). It is shown that the SDSI has two separate layers of complexity that can be captured in a precise way by a pair of functional Feynman-Kac equations for the Laplace transform. Exact solutions are obtained as power series expansions in the intermittency parameter using a novel intermittency differentiation rule. The expansion of the moments gives a new representation of the Selberg integral.

KEY WORDS: multifractals, intermittency, SDSI, Feynman-Kac, long-range dependence, Selberg integral

1. INTRODUCTION

Multifractal stochastic processes provide an important new mathematical tool for modelling stochastic phenomena that exhibit long-range dependence and self-similarity. Such phenomena arise empirically in many areas of science ranging from the physics of turbulence(24,30) to geophysics(29) to human heartbeat dynamics and physiology(13,14) to asset returns.(6,33) While grid-bound canonical multifractals of Mandelbrot(21) have long been extant, the first formal constructions of grid-free multifractal processes have only been given very recently,

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confer.[2,5,7,26,31] All of these processes belong to the same class that we will refer to as Limit Log-Infinitely Divisible multifractals.

Limit Log-Infinitely Divisible multifractals are constructed by replacing the canonical hierarchy of products of independent identically distributed random weights with a hierarchy of stochastic integrals over conical domains in the time-scale plane and then taking the zero scale limit. The limit processes are thus scale-free and have dependent, non-gaussian, and stationary increments. Most importantly, they possess many remarkable features such as nonlinear moment scaling, *i.e.* multiscaling, and stochastic self-similarity with log-infinitely divisible multipliers, also known as scale-consistent continuous dilation invariance, which make them into ideal candidates for multiplicative noises.

The subclass of limit lognormal multifractals was introduced and reviewed by Mandelbrot,[19,22] and formalized in a series of papers by Kahane.[15–17] An original construction of limit lognormal multifractals as stochastic processes as opposed to random measures first appeared in Bacry *et al.*,[2] who investigated many of the properties of the limit process without rigorously taking the zero scale limit. Most recently, a novel approach to limit lognormal multifractals was presented in Schmitt[32] (henceforth referred to as *the* Schmitt process), whose construction is based on stochastic integrals with respect to Brownian motion as opposed to 2D gaussian random measures and is thereby simplified compared to the previous work.

In this paper we continue our study of limit lognormal multifractals. In our previous work[28] we quantified the distribution of the Bacry *et al.*[2] process (henceforth referred to as *the* limit lognormal process) by a pair of exact integro-differential relations for the Laplace transform and interpreted them as multifractal counterparts of the star equation of Mandelbrot[20] and of the classical Girsanov theorem.[12] These relations were obtained as the zero scale limit of discrete hierarchies of finite scale approximations to the limit process. In this paper we will introduce a more powerful continuous time approach that does not require any discretization. It is based on a combination of the backward Kolmogorov equation, a generalized Girsanov-type theorem for gaussian processes, and, most importantly, new invariance properties of the limit lognormal process that are stated and proved in the paper. This enables us to derive new functional Feynman-Kac equations for the Laplace transform of the process.

The classical Feynman-Kac formula is a parabolic partial differential equation for the Laplace transform of certain functionals of Brownian motion and, more generally, Ito diffusions driven by it. The origin of the classical formula is the strong Markov property of these diffusions. Our main contribution is that the Laplace transform of the limit lognormal process, which is defined as the exponential functional of a strongly *nonmarkovian* gaussian process, can still be described by generalized Feynman-Kac equations. Our approach is based on invariance properties of this gaussian process and does recover the classical formula if we replace the gaussian process with Brownian motion. Hence the name 'Feynman-Kac.'

The new equations improve over the previous results in four ways. First, they capture and help distinguish in a precise way between the two separate layers of complexity of the stochastic dependence structure of increments (SDSI) of the limit lognormal process. Indeed, as this paper illustrates using the Schmitt process as an example, the limit lognormal SDSI has two layers of complexity that can be separated from each other. By removing the secondary and keeping the primary layer, we obtain a toy limit lognormal process that has the same basic structure of the SDSI. In addition, the new process shares some of the main properties of the limit lognormal and Schmitt processes such as stationary increments and a finite decorrelation length. We derive two pairs of functional Feynman-Kac equations, each pair consisting of one equation for the exact and the other for the toy process. The equations in the first pair exhibit the same basic mathematical structure thereby attesting to the fact that the processes possess the same primarily complexity layer of the SDSI. The equations in the second pair show different mathematical structures. It is this difference that encodes the secondary layer of complexity that both the limit lognormal and Schmitt processes have and the toy model does not. We will collectively refer to all these equations as the SDSI equations.

The SDSI equations are a simpler set of equations than the generalized star equation derived in Ref. 28. While both approaches involve integrals with respect to the entire path of the limit lognormal process, the former requires only one such integral while the latter a whole continuum. However, their principal limitations are the same, namely, that they are not single-variable equations due to the path integral terms they contain. Remarkably, the second Feynman-Kac equation for the toy process does reduce to a single-variable equation and can be used to actually compute the underlying distribution.

The second contribution of the paper is the development of an exact expansion of the Laplace transform of the limit lognormal distribution in powers of the intermittency parameter. This expansion is a corollary of yet another functional Feynman-Kac formula that is derived in the paper, which we think of as the intermittency differentiation rule. While this formula also contains path dependent terms, its main difference from the SDSI equations is that it can be iterated ad infinitum. The coefficients of the expansion are computed recursively by multiple integration. We believe this expansion to be the first result towards the goal of explicitly computing the limit lognormal distribution and as such a major advance in the field of multifractal modeling. Moreover, the differentiation rule allows us to obtain similar power series expansions for all path dependent terms that enter the SDSI equations as well as the generalized star equation, thereby formally solving them exactly. The actual computation of all expansion coefficients is left for further research, we do, however, compute the first nontrivial terms in the intermittency parameter and verify that all the equations hold in this approximation.

Third, the invariance properties of the limit lognormal process that are introduced in the paper lead to simpler proofs of several of our previous results that were originally established in Ref. 28 using a finite scale analysis.

Fourth, the SDSI equations for the limit lognormal process imply interesting recurrence relations for the Selberg integral,[34] which is known to represent its moments, confer.[3] In addition, the intermittency differentiation rule implies a novel representation of the Selberg integral as a power series in the intermittency parameter.

The plan of the paper is as follows. In Sec. 2 we present a brief review of the limit lognormal and Schmitt processes and show that the limit lognormal distribution arises from a noncentral limit theorem. In Sec. 3 we describe the SDSI of the Schmitt process and explain that it can be thought of as consisting of two separate layers of complexity. We introduce a toy limit lognormal model that captures the first layer only. In Sec. 4 we introduce our technique of functional Feynman-Kac equations and state the first and second pairs of equations. In Sec. 5 we state the intermittency differentiation rule, study the ensuing recurrence relations, and derive infinite series representations for the Laplace transform of the limit lognormal process, its moments, and related functionals. In Sec. 6 we give a summary of the most significant results that appeared first in Ref. 28, clarify the significance of the decorrelation length, and state the Selberg integral recurrences. In Sec. 7 we treat the low intermittency limit. In Sec. 8 we give proofs of the main results. Section 9 presents conclusions. The Appendix describes the law of the toy model.

2. LIMIT LOGNORMAL MULTIFRACTALS

In this section we will give a concise summary of limit lognormal multifractals following Refs. 26 and 32. We start by describing their properties, then proceed to the constructive definition, and end the section with an original reformulation of the limit lognormal distribution in light of a noncentral limit theorem. For the sake of brevity, we will restrict our attention to increasing and positive processes known as random times. This restriction can be easily relaxed by considering Brownian motion in such time instead.

In the broadest sense, a limit lognormal random time is an increasing and positive stochastic process $M(t)$, whose increments, denoted by $\delta_l M(t) \equiv M(t+l) - M(t)$, obey the following multiscaling law in the limit of small increment size

$$\mathbf{E}\left[\delta_l M(t)^q\right] \propto l^{\zeta(q)}, \quad l \to 0, \tag{1}$$

$$\zeta(q) = q\Big(1 + \frac{\mu}{2}\Big) - \frac{\mu q^2}{2}. \tag{2}$$

The function $\zeta(q)$ is known as the multifractal spectrum and dates back to the work of Mandelbrot.(22) The meaning of Eq. (1) is that the moments of the increments of $M(t)$ behave as a power law of the increment size, and the exponent is a nonlinear function of the moment q, hence the 'multi' qualifier.

We are aware of two classes of explicit constructions that give rise to Eq. (1). Both are based on the idea of considering exponential functionals of stationary gaussian processes that fall under the class of special T-martingale processes investigated by Kahane in a series of papers.(15–17) Specifically, let $\omega_\varepsilon(s)$ be a stationary gaussian process in s, whose mean and covariance are functions of ε that plays the role of a finite scale. We consider the process

$$M_\varepsilon(t) = \int_0^t e^{\omega_\varepsilon(s)}\, ds. \tag{3}$$

The limit lognormal construction, due to Bacry *et al.*,(2) defines the mean and covariance of $\omega_\varepsilon(s) \equiv \omega_{\mu,L,\varepsilon}(s)$ to be

$$\mathbf{E}[\omega_\varepsilon(t)] = -\frac{\mu}{2}\left(1 + \log\frac{L}{\varepsilon}\right), \tag{4}$$

$$\mathbf{Cov}[\omega_\varepsilon(t),\ \omega_\varepsilon(s)] = \mu\ \log\frac{L}{|t-s|}, \qquad \varepsilon \le |t-s| \le L, \tag{5}$$

$$\mathbf{Cov}[\omega_\varepsilon(t),\ \omega_\varepsilon(s)] = \mu\left(1 + \log\frac{L}{\varepsilon} - \frac{|t-s|}{\varepsilon}\right), \tag{6}$$

if $|t-s| < \varepsilon$, and covariance is zero in the remaining case of $|t-s| \ge L$. Thus, ε is used as a truncation scale. L is the fundamental decorrelation length of the process that regulates the extent of long-range dependence. Indeed, nonoverlapping increments of $M_\varepsilon(t)$ are dependent only if they are within L apart. μ is the intermittency parameter.[3] Note that $\mathbf{E}[\exp(\omega_\varepsilon(s))] = 1$ so that $\mathbf{E}[M_\varepsilon(t)] = t$.

The second construction is due to Schmitt,(32) which we have re-cast here into a form that is compatible with the first construction. Let $\omega_\varepsilon(s)$ be defined by

$$\omega_\varepsilon(s) = \sqrt{\mu}\int_{s+\varepsilon}^{L+s} \frac{1}{\sqrt{u-s}}\, dB(u) - \frac{\mu}{2}\log\frac{L}{\varepsilon}. \tag{7}$$

$B(u)$ denotes the standard one-dimensional Brownian motion starting at zero. The roles that ε, L, and μ play are exactly the same as what they are in the limit lognormal construction. In particular, $\omega_\varepsilon(s)$ is stationary, the covariance between $\omega_\varepsilon(s)$ and $\omega_\varepsilon(t)$ is nonzero only so long as $|t-s| < L$, and $\mathbf{E}[\exp(\omega_\varepsilon(s))] = 1$.

[3] It is more common to use λ^2 instead of μ and refer to λ as the intermittency parameter. However, our results are most naturally expressed in terms of μ.

The covariance is calculated to be

$$\mathbf{Cov}\,[\omega_\varepsilon(t),\ \omega_\varepsilon(s)] = 2\mu\ \log\frac{\sqrt{L-|t-s|}+\sqrt{L}}{\sqrt{|t-s|+\varepsilon}+\sqrt{\varepsilon}},\qquad |t-s|\le L. \tag{8}$$

The interest in both the limit lognormal and Schmitt constructions stems from the $\varepsilon \to 0$ limit. Bacry and Muzy[4] showed that $M_\varepsilon(t)$ converges weakly (as a measure on $\mathbb{R}^+$) a.s. to a limit process $M(t) = \lim_{\varepsilon\to 0} M_\varepsilon(t)$ provided $0 \le \mu < 1$, the limit is nondegenerate in the sense that $\mathbf{E}[M(t)] = t$, and, most importantly, obeys Eq. (1). It is not too difficult to verify that the same is true of the Schmitt construction. The condition for the finiteness of the positive moments of both processes is

$$\mathbf{E}\,[M(t)^q] < \infty\ \Leftrightarrow q < \frac{2}{\mu}. \tag{9}$$

In particular, the moments do become infinite so that the naive expansion of the Laplace transform of $M(L)$ in terms of the moments is not possible. In both cases, the existence and nondegeneracy of the limit is based on the theory of convergence of T-martingales of Kahane.[16] The fundamental reason for multifractality in the sense of Eq. (1) is that in both constructions the covariance blows up as

$$\mathbf{Cov}\,[\omega_\varepsilon(t),\ \omega_\varepsilon(s)] \propto \log|s-t| \tag{10}$$

in the limit $|s-t| \to 0$. This particular asymptotic behavior and the condition $\mathbf{E}[\exp(\omega_\varepsilon(s))] = 1$ are known from the general theory of lognormal multiplicative chaos of Kahane[15] to be sufficient for generating multifractality.

The limit lognormal process possesses a much stronger form of multifractality than Eq. (1) known as stochastic self-similarity or scale-consistent continuous dilation invariance,

$$M(t) = W_{t/L}\,M(L),\ \ t < L, \tag{11}$$

understood as the equality in law. $W_{t/L}$ is a lognormal multiplier that is independent of $M(L)$, confer Ref. 26 for the original derivation or Ref. 28 for review. For our purposes, the main implication of Eq. (11) is that the law of $M(L)$ determines that of $M(t)$ for any $t < L$ so that we need only understand $M(L)$.

We will now reformulate the limit distribution as an explicit noncentral limit problem. Let us break up time into the subintervals of length ε so that $s_j = j\varepsilon$, $\omega_j \triangleq \omega_\varepsilon(s_j)$, and $N\varepsilon = L$. It is shown in Ref. 4 that the limit distribution $M(L)$ can be approximated as

$$\varepsilon\sum_{j=0}^{N-1} e^{\omega_j} \to M(L)\ \text{as}\ \varepsilon\to 0. \tag{12}$$

Now, consider a new set of random variables η_j defined by

$$\eta_j = \omega_j - \omega_{j-1}, \quad j = 1 \ldots N-1. \tag{13}$$

Clearly, the sum in Eq. (12) can be written as

$$\mathrm{e}^{\omega_0} \frac{L}{N} \sum_{j=0}^{N-1} \prod_{k=1}^{j} \mathrm{e}^{\eta_k} \equiv \mathrm{e}^{\omega_0} \frac{L}{N} \big(1 + \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_1}\mathrm{e}^{\eta_2} + \cdots + \mathrm{e}^{\eta_1} \cdots \mathrm{e}^{\eta_{N-1}}\big). \tag{14}$$

What is remarkable about the representation in Eq. (14) is that the η_j are *renormalized*. Indeed, it is easy to see using Eqs. (4)–(6) that the joint distribution of the η_j and ω_0 is gaussian with the following means and variances[4]

$$\mathbf{E}[\eta_j] = 0, \quad \mathbf{Var}[\eta_j] = 2\mu, \tag{15}$$

$$\mathbf{Cov}[\eta_j,\ \eta_{j+l}] = \mu\big(\log(l+1) + \log(l-1) - 2\log(l)\big), \tag{16}$$

$$\mathbf{Cov}[\omega_0,\ \eta_j] = \mu\big(\log(j-1) - \log(j)\big). \tag{17}$$

It follows that the joint distribution of the η_j depends on N only in the their total number so that the only nontrivial N dependence in Eq. (14) comes from the prefactor e^{ω_0}/N. Thus, the problem of computing the $N \to \infty$ limit of the distribution that appears in Eq. (14) is formally similar to the sum of products problems that naturally appear in the physics of 1D disordered systems, confer Refs. 9 and 18 for example. However, the fundamental difference between the two is that in our case the random factors $\exp(\eta_j)$ are strongly dependent unlike in disordered system problems. This strong dependence is the main source of complexity of as well as interest in the limit lognormal construction.

This completes our overview of limit lognormal multifractals. The main purpose of this paper is to quantify the limiting distribution of the limit lognormal process $M(t)$ at $t = L$ beyond what we did in Ref. 28. The essential complexity of the task has to do with the very complicated stochastic dependence structure of increments (SDSI) of $M(t)$, the subject that we proceed to next.

3. STOCHASTIC DEPENDENCE STRUCTURE OF INCREMENTS (SDSI)

In this section we will explain the SDSI of limit lognormal processes. We explained in Ref. 28 that the main properties of this structure are: first, all nonoverlapping increments of $M(t)$ that are within the distance L apart are dependent and, second, their dependence requires infinitely many independent multipliers. The Schmitt construction enables us to further elucidate the SDSI of $M(t)$ due to the availability

[4] Using the convention that $\log(0) = -1$.

of an explicit representation of $\omega_\varepsilon(s)$ in Eq. (7) in terms of Brownian motion. While the SDSI of the limit lognormal process has the same mathematical structure, it is more difficult to extract as we do not know of any explicit representation of $\omega_\varepsilon(s)$ in Eqs. (4)–(6) in terms of Brownian motion.

We start with Eq. (7) and discretize it by breaking up time into subintervals of length ε each and then replacing integrals with sums. Denote $f(u) = 1/\sqrt{u}$. The natural discrete time approximation to $M_\varepsilon(t)$ is

$$M_\varepsilon(t) \approx \varepsilon \left(\frac{L}{\varepsilon}\right)^{-\frac{\mu}{2}} \sum_{s=0}^{t} \prod_{u=s+\varepsilon}^{L+s} \exp\left(\sqrt{\mu} f(u-s) \Delta B(u)\right). \tag{18}$$

Each individual summand in Eq. (18) contains the product of *independent* lognormal random weights $\exp(\sqrt{\mu} f(u-s)\Delta B(u))$ as nonoverlapping increments of Brownian motion $\Delta B(u)$ are independent. However, the weights are *recurrent* in the sense that the same weights occur for different values of s. Thus, in the limit $\varepsilon \to 0$, we end up with the infinite sum of infinite products of these recurrent weights. This is the origin of the SDSI of the process.

The mathematical complexity of the problem is twofold. We believe that it is determined primarily by the recurrence of weights. The secondary layer of complexity is that the weights are not identically distributed, *i.e.* $f(u)$ is not a constant. It is precisely the singularity of $f(u)$ at the origin that is responsible for the logarithmic singularity in the covariance in Eq. (10), thus giving rise to multifractality. However, if we make $f(u)$ constant and thereby give up multifractality, the primary source of complexity remains intact.

This observation leads us to consider a new process, stripped of the secondary layer of complexity by setting $f(u) = 1$, which has the effect of making the lognormal weights in Eq. (18) to be all identically distributed and results in

$$M^{(\mathrm{toy})}(t) = \int_0^t \mathrm{e}^{\sqrt{\mu}\left(B(L+s)-B(s)\right)-\mu\frac{L}{2}}\, ds. \tag{19}$$

We call it the toy limit lognormal process. $M^{(\mathrm{toy})}(t)$ shares most of the features of the exact limit lognormal process, namely, it has stationary increments that are dependent only so long as they are within L apart, *i.e.* L continues to play the role of the decorrelation length of the process, and $\mathbf{E}[M^{(\mathrm{toy})}(t)] = t$. The main difference is that $M^{(\mathrm{toy})}(t)$ is not multifractal. Its law is derived in the Appendix.

In the next section we will state two pairs of functional Feynman-Kac equations that quantify the primary and secondary complexity layers, respectively, which we introduced in this section.

4. FEYNMAN-KAC EQUATIONS FOR THE TWO LAYERS OF COMPLEXITY

As is well-known, the Feynman-Kac formula for the standard one-dimensional Brownian motion starting at x is the following partial differential equation for the functional $u(x,\, L) = \mathbf{E}^x[\exp(-\int_0^L q(B(u))\,du)]$, where $q(x)$ is a smooth, positive function,

$$\frac{1}{2}\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial L} + q(x)u. \tag{20}$$

In the special case of $q(x) = z\exp(x)$ and $x = 0$, we then obtain the Feynman-Kac formula for the Laplace transform of the exponential functional of Brownian motion starting at zero $v(z,\, L) = \mathbf{E}[\exp(-z\int_0^L e^{B(u)}du)]$

$$\frac{1}{2}\left[z\frac{\partial v}{\partial z} + z^2\frac{\partial^2 v}{\partial z^2}\right] = \frac{\partial v}{\partial L} + zv(z,\, L). \tag{21}$$

Our goal in this section is to derive the analogues of Eq. (21) for the Laplace transform of the exact and toy limit lognormal processes and thereby quantify their laws and capture the primary and secondary complexity layers of their SDSIs. Before we proceed to state our results, it is worth pointing out that the essential novelty as well as difficulty of the problem is that the gaussian processes that enter the exponential functionals in Eqs. (3) and (19) are *non-markovian*.

Our main object of interest throughout this paper is the following generalized Laplace transform

$$v(z,\, \mu,\, L,\, f) \triangleq \mathbf{E}\left[\exp\left(-z\int_0^L e^{\mu f(s)}\,dM(s)\right)\right] \tag{22}$$

of the limit lognormal process, where $f(s)$ is an arbitrary smooth function[5] that may depend on L but does not involve the intermittency parameter μ. All $f(s)$ functions that appear in applications below are of the form $f(s) = g(s,\, u)$ for some $u \in [0,\, L]$, and the function $g(s_1,\, s_2)$ is defined by

$$g(s_1,\, s_2) \triangleq \log\frac{L}{|s_1 - s_2|}. \tag{23}$$

Its significance is that $\lim_{\varepsilon\to 0}\mathbf{Cov}(\omega_\varepsilon(s_1), \omega_\varepsilon(s_2)) = \mu\, g(s_1, s_2)$. The integration with respect to the limit measure $dM(s)$ is understood in the sense of $\varepsilon \to 0$ limit of $dM_\varepsilon(s)$ so that $v(z,\, L,\, f) = \lim_{\varepsilon\to 0} v_\varepsilon(z,\, L,\, f)$, where $v_\varepsilon(z,\, L,\, f)$ is the generalized Laplace transform of the random measure $dM_\varepsilon(s)$ introduced in

[5] Unless stated otherwise, the term 'function' is reserved for nonrandom functions. Throughout this section we fix $0 \le \mu < 1$ and drop it from the list of arguments of the generalized Laplace transform to abbreviate $v(z,\, L,\, f)$.

Eq. (3) with the $\omega_\varepsilon(s)$ process as in Eqs. (4)–(6). In the special case of $f \equiv 0$, $v(z, L, f)$ is the Laplace transform of $M(L)$, which we will denote by $v(z, L)$.

Proposition 4.1. *The generalized Laplace transform $v(z, L, f)$ of the limit lognormal process satisfies*

$$\left(\frac{\mu}{2L}z^2\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial L}\right)v(z, L, f) = z\, e^{\mu f(L)}\, v\big(z, L, f + g(\cdot, L)\big) + \mu z \int_0^L \frac{\partial f}{\partial L}(u)\, e^{\mu f(u)}\, v\big(z, L, f+g(\cdot, u)\big)\, du. \tag{24}$$

Corollary 4.1. *The Laplace transform $v(z, L)$ of the limit lognormal distribution $M(L)$ satisfies*

$$\left(\frac{\mu}{2L}z^2\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial L}\right)v(z, L) = z\, v\big(z, L, g(\cdot, L)\big). \tag{25}$$

This is the first Feynman-Kac formula for the limit lognormal process.

Corollary 4.2. *The Laplace transform $v(z, L)$ of the limit lognormal distribution $M(L)$ satisfies*

$$\left(\frac{\mu}{2L}z^2\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial L}\right)^2 v(z, L) = z^2\, v\big(z, L, g(\cdot, 0) + g(\cdot, L)\big). \tag{26}$$

This is the second Feynman-Kac formula for the limit lognormal process. The proof of Proposition 4.1 and its corollaries is deferred to Sec. 8 so as not to interrupt the flow of presentation in this section.

We will now contrast these results with the corresponding results for the toy limit lognormal process. Let the generalized Laplace transform be denoted by $v^{(\text{toy})}(z, L, f) \triangleq \mathbf{E}[\exp(-z\int_0^L e^{\mu f(s)}\, dM^{(\text{toy})}(s))]$. The covariance of the $\omega(s) \triangleq \sqrt{\mu}(B(L+s) - B(s)) - \mu L/2$ process in Eq. (19) is $\mu(L - |s-t|)$ so that $g^{(\text{toy})}(s_1, s_2) \triangleq L - |s_1 - s_2|$.

Proposition 4.2. *The Laplace transform $v^{(\text{toy})}(z, L)$ of the toy limit lognormal distribution satisfies*

$$\left(\frac{\mu}{2}z^2\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial L}\right)v^{(\text{toy})}(z, L) = z\, v^{(\text{toy})}\big(z, L, g^{(\text{toy})}(\cdot, L)\big), \tag{27}$$

$$\left(\frac{\mu}{2}z^2\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial L}\right)^2 v^{(\text{toy})}(z, L) = z^2\, v^{(\text{toy})}(ze^{\mu L}, L). \tag{28}$$

These are the Feynman-Kac formulas for the toy limit lognormal process. The proof is in Sec. 8.

We can draw several conclusions from Propositions 4.1 and 4.2. First, we see a clear similarity between Eqs. (25) and (27). They possess the same general mathematical structure, and their right-hand sides exhibit the same type of functional shifts that are induced by the corresponding covariances *via* the g functions. This similarity is indicative of the fact that the pair of Eqs. (25) and (27) captures the same primary layer of complexity of the exact and toy limit lognormal SDSIs.

Second, Eqs. (25) and (27) are essentially different from the classical Feynman-Kac formula in Eq. (21). While all three equations are of parabolic type, Eq. (21) is a regular partial differential equation, whereas Eqs. (25) and (27) are functional differential equations as they involve the entire path of the process up to time L. This is reflective of the fact that our problem is intrinsically non-markovian.

Third, what is not immediately obvious is that despite the difference between the classical Feynman-Kac formula in Eq. (21) and the functional Feynman-Kac formulas in Eq. (25) and (27), they have the same mathematical origin as shown in Sec. 8, hence the 'Feynman-Kac' appellation.

Finally, the pair of Eqs. (26) and (28) distinguishes between the secondary layers of complexity of the exact and toy models, *i.e.* the presence of non-identically distributed weights in the exact construction. As did the first pair, Eqs. (26) and (28) have the same mathematical structure. Indeed, their left-hand sides involve squares of parabolic operators and their right-hand sides involve the same type of functional shifts. This follows from the identity $g^{(\text{toy})}(s, 0) + g^{(\text{toy})}(s, L) = L$ so that $v^{(\text{toy})}(ze^{\mu L}, L) = v^{(\text{toy})}(z, L, g^{(\text{toy})}(\cdot, 0) + g^{(\text{toy})}(\cdot, L))$. The difference between Eqs. (26) and (28) is also manifest, for Eq. (26) is fully path-dependent, whereas Eq. (28) is a single-variable equation for the dependent variable $v^{(\text{toy})}(z, L)$, due to the cancelation of s dependence. The persistence of this path-dependence is what distinguishes between the secondary layers of complexity. In summary, it is the difference in fine properties of the underlying covariance structures that leads to the different secondary layers of complexity.

The same point can be argued by noticing that the presence of a path dependent term in Eq. (26) and its lack in Eq. (28) is indicative of how "far" the underlying $\omega(s)$ process is from being markovian. Indeed, it is not difficult to see that in the toy model this process can be represented as the sum of a Brownian motion and an independent time-reversed Brownian motion *via* $B(L+s) - B(s) = (B(L+s) - B(L)) + (B(L) - B(s))$, which is "almost" markovian. It is this property that enables us to compute the law of $M^{(\text{toy})}(L)$, confer the Appendix.

Henceforth, we will refer to Eqs. (25) and (26) as the SDSI equations for the limit lognormal process.

5. A FEYNMAN-KAC EQUATION FOR THE INTERMITTENCY DIFFERENTIATION

In this section we will establish a differentiation rule that quantifies how the distribution of the limit lognormal process varies with the intermittency parameter. Mathematically, it can be thought of as another Feynman-Kac formula that is obeyed by the Laplace transform of the process. The principal difference of this result from those of the preceding section is that the differentiation formula can be iterated ad infinitum, thus resulting in power series expansions for the Laplace transform as well as the moments of the limit lognormal process.

As in Sec. 4, the main object of study is the generalized Laplace transform $v(z,\, \mu,\, L,\, f)$. Throughout this section we fix L and drop it from the list of arguments so as to write $v(z,\, \mu,\, f)$ instead. The following theorem quantifies how $v(z,\, \mu,\, f)$ varies as a function of μ.

Proposition 5.1. *The generalized Laplace transform $v(z,\, \mu,\, f)$ solves*

$$\frac{\partial}{\partial\mu} v(z,\, \mu,\, f) = \frac{z^2}{2}\int_{\otimes 2} v\big(z,\, \mu,\, f + g(\cdot,\, s_1) + g(\cdot,\, s_2)\big) e^{\mu(f(s_1)+f(s_2)+g(s_1,\, s_2))} \times g(s_1,\, s_2)\, d\vec{\mathbf{s}}^{(2)} - z\int_{\otimes 1} v\big(z,\, \mu,\, f + g(\cdot,\, s)\big) e^{\mu f(s)} f(s)\, ds, \quad (29)$$

where $\int_{\otimes k}$ is an abbreviation for $\int_0^L \cdots \int_0^L d\vec{\mathbf{s}}^{(k)}$ and $g(s_1, s_2)$ is as in Eq. (23).

The proof is given in Sec. 8. We can summarize Eq. (29) by saying that differentiation with respect to the intermittency parameter μ is equivalent to a combination of two functional shifts induced by the g function. The first corresponds to the dependence of $M(t)$ on μ, while the second to the appearance of μ in the exponential integrand in Eq. (22).

Corollary 5.1. *The generalized Laplace transform $v(z,\, \mu,\, f)$ satisfies*

$$\frac{\partial^n}{\partial\mu^n} v(z,\, \mu,\, f) = \sum_{k=1}^{2n} z^k \int_{\otimes k} v\left(z,\, \mu,\, f + \sum_{i=1}^{k} g(\cdot,\, s_i)\right) e^{\mu\left(\sum_{i<j}^{k} g(s_i, s_j) + \sum_{i=1}^{k} f(s_i)\right)} \times h_{n,k}(\vec{\mathbf{s}})\, d\vec{\mathbf{s}}^{(k)}. \quad (30)$$

for some functions $h_{n,k}(\vec{\mathbf{s}}) \equiv h_{n,k}(s_1, \ldots, s_k)$, $k = 1 \ldots 2n$, that are computed iteratively via the following three-term recurrence[6]

$$h_{n+1,k}(\vec{\mathbf{s}}) = \frac{1}{2} h_{n,k-2}(\vec{\mathbf{s}}) g(s_{k-1}, s_k) - h_{n,k-1}(\vec{\mathbf{s}}) \left(\sum_{i=1}^{k-1} g(s_i, s_k) + f(s_k) \right)$$
$$+ h_{n,k}(\vec{\mathbf{s}}) \left(\sum_{i<j}^{k} g(s_i, s_j) + \sum_{i=1}^{k} f(s_i) \right), \quad k = 1 \ldots 2n+2, \tag{31}$$

starting with

$$h_{1,1}(s) = -f(s), \quad h_{1,2}(s_1, s_2) = \frac{1}{2} g(s_1, s_2). \tag{32}$$

The functions $h_{n,k}(\vec{\mathbf{s}})$ can be taken to be symmetric in $s_1 \cdots s_k$.

Several remarks are in order. First, the recurrence relation of Eq. (31) does not automatically produce a symmetric $h_{n+1,k}(\vec{\mathbf{s}})$, however the symmetry can always be imposed as the rest of the integrand in Eq. (30) is manifestly symmetric. Second, if $f \equiv 0$, then the range of k is changed to $2 \cdots 2n$. Finally, the proof of Corollary 4.1 is a straightforward, while tedious, application of the intermittency differentiation rule of Proposition 5.1.

It is worth listing the first few $h_{n,k}(\vec{\mathbf{s}})$ functions so as to clarify their general structure. For simplicity, we assume that $f \equiv 0$. Then, after symmetrization, we have

$$h_{2,2}(s_1, s_2) = \frac{1}{2} g^2(s_1, s_2), \tag{33}$$

$$h_{2,3} = -\frac{1}{3} \big(g(s_1, s_2) g(s_1, s_3) + g(s_2, s_1) g(s_2, s_3) + g(s_3, s_1) g(s_3, s_2) \big), \tag{34}$$

$$h_{2,4} = \frac{1}{12} \big(g(s_1, s_2) g(s_3, s_4) + g(s_1, s_3) g(s_2, s_4) + g(s_1, s_4) g(s_2, s_3) \big). \tag{35}$$

Recalling the definition of the g function in Eq. (23), we can conclude that the $h_{n,k}$ functions are formed as symmetrized products of logarithms. Moreover, it is easy to see that each $h_{n,k}$ is a sum of products, each of which having exactly n factors of the g function. It follows that in the case of $f \equiv 0$, the integrals

$$H_{n,k} \equiv \int_0^1 \cdots \int_0^1 h_{n,k}(\vec{\mathbf{t}} L)\, d\vec{\mathbf{t}}^{(k)} \tag{36}$$

are independent of L so that $\int_{\otimes k} h_{n,k}(\vec{\mathbf{s}})\, d\vec{\mathbf{s}}^{(k)} = L^k H_{n,k}$.

[6] Empty sums and $h_{n,k}(\vec{\mathbf{s}})$ for the values of k outside of $k = 1 \cdots 2n$ are understood to mean zero.

Corollary 5.1 yields the Taylor expansion of the generalized Laplace transform in powers of the intermittency parameter as its immediate corollary. Indeed, if $\mu = 0$, $M(t) = t$ so that all the higher derivatives can be computed iteratively.

Corollary 5.2.

$$\mathbf{E}\left[\exp\left(-z\int_0^L e^{\mu f(s)}\,dM(s)\right)\right] = e^{-zL}\left(1+\sum_{n=1}^{\infty}\frac{\mu^n}{n!}\left[\sum_{k=1}^{2n} z^k \int_{\otimes k} h_{n,k}(\vec{\mathbf{s}})\,d\vec{\mathbf{s}}^{(k)}\right]\right). \tag{37}$$

Thus, we have effectively shown how to compute the Laplace transform of $\int_0^L e^{\mu f(s)}\,dM(s)$ for an arbitrary $f(s)$ that does not itself involve μ. In particular, all functionals that occur in Proposition 4.1 and its corollaries are of this form so that Eq. (37) gives exact solutions to the SDSI equations.

For the rest of this section we restrict ourselves to the case of the Laplace transform of $M(L)$ that corresponds to $f \equiv 0$ and the k sum in Eq. (37) starting at $k = 2$. We obtain

$$\mathbf{E}\left[\mathrm{e}^{-zM(L)}\right] = \mathrm{e}^{-zL}\left(1+\sum_{n=1}^{\infty}\frac{\mu^n}{n!}\left[\sum_{k=2}^{2n}(zL)^k H_{n,k}\right]\right). \tag{38}$$

As expected, confer[28] and Sec. 6 below, the Laplace transform is a function of μ and zL.

We conclude this section with an application of Corollary 5.2 to the Selberg integral. The integral moments of the limit lognormal process were shown in Ref. 3 to be given by the celebrated Selberg integral, confer Refs. 1 and 34

$$\mathbf{E}[M(L)^m] = L^m \int_0^1 \cdots \int_0^1 \prod_{i<j}^{m} |s_i - s_j|^{-\mu} d\vec{\mathbf{s}}^{(m)}, \quad m \geq 2, \tag{39}$$

which is convergent provided $m < 2/\mu$, confer Eq. (9).

Proposition 5.2. *Given $0 \leq \mu < 1$ and $2 \leq m < 2/\mu$, the Selberg integral has the expansion*

$$\int_0^1 \cdots \int_0^1 \prod_{i<j}^{m} |s_i - s_j|^{-\mu} d\vec{\mathbf{s}}^{(m)} = 1 + \sum_{n=1}^{\infty}\frac{\mu^n}{n!}\sum_{k=2}^{\min\{2n,m\}}(-1)^k \frac{m!}{(m-k)!} H_{n,k}. \tag{40}$$

The proof is deferred to Sec. 8.

In summary, Eq. (37) gives us the complete expansion of the generalized Laplace transform of the limit lognormal process and Eq. (40) of the Selberg integral in powers of μ. In Sec. 7 we will explicitly compute these expansions up to the first nontrivial terms in the intermittency parameter μ and verify the SDSI equations to this order of accuracy.

6. RESULTS OF FINITE SCALE ANALYSIS REVISITED

In this section we will give a summary of the main results that appeared first in Ref. 28 and clarify the role of the decorrelation length L. The original arguments were based on discretization and the resulting finite scale analysis. We can now give new proofs that do not require discretization and follow the pattern of the two preceding sections. This is done in Sec. 8.

The primary object of interest in this section is the Laplace transform of $M(t),\ t < L$, that we denote by $v(z,\ \mu,\ t,\ L)$. As before, $v(z,\ \mu,\ L)$ denotes the Laplace transform of $M(L)$ so that $v(z,\ \mu,\ L) = v(z,\ \mu,\ L,\ L)$. We also need the family of auxiliary processes $M^{(u)}(t),\ 0 \le u \le 1$, that we first introduced in Ref. 28

$$M^{(u)}(t) = \int_0^t \left|\frac{s}{t} - u\right|^{-\mu} dM(s). \tag{41}$$

It is worth noting that these processes are multifractal in the sense of Eq. (11) as shown in Ref. 28. Clearly, $M^{(u=0)}(L)$ is the auxiliary random variable that appeared in Corollary 4.1.

Proposition 6.1. *The Laplace transform $v(z,\ \mu,\ t,\ L)$ solves*

$$\left(t\frac{\partial}{\partial t} + \frac{\mu}{2}z^2\frac{\partial^2}{\partial z^2} - z\frac{\partial}{\partial z}\right)v(z,\ \mu,\ t,\ L) = 0. \tag{42}$$

Equation (42) was shown in Ref. 28 to be a statement of stochastic self-similarity of $M(t)$ and is equivalent to Eq. (11). In particular, it does not capture the distribution of $M(L)$ but only of $M(t),\ t < L$, in terms of $M(L)$.

Proposition 6.2. *The Laplace transform $v(z,\ \mu,\ t,\ L),\ t < L$, solves*

$$\left(t\frac{\partial}{\partial t} + L\frac{\partial}{\partial L} - z\frac{\partial}{\partial z}\right)v(z,\ \mu,\ t,\ L) = 0, \tag{43}$$

$$z\frac{\partial}{\partial z}v(z,\ \mu,\ L) = L\frac{\partial}{\partial L}v(z,\ \mu,\ L). \tag{44}$$

Equation (43) means that $v(z,\ \mu,\ t,\ L)$ is a function of $(\mu,\ zL,\ t/L)$ and Eq. (44) means that $v(z,\ \mu,\ L)$ is a function of μ and zL. Both assertions, but not the equations, appeared first in Ref. 28.

Proposition 6.3. *The Laplace transform $v(z,\ \mu,\ t,\ L)$, $t < L$, solves*

$$\frac{\partial}{\partial z} v(z,\ \mu,\ t,\ L) = -t \int_0^1 \mathbf{E}\big[\mathrm{e}^{-z(\frac{t}{L})^{-\mu} M^{(u)}(t)}\big]\, du. \tag{45}$$

Equation (45) was interpreted in Ref. 28 as a generalized star equation of Mandelbrot.

In light of Eq. (44), it is sufficient to restrict ourselves to $L = 1$ due to $v(z,\ \mu,\ L) = v(zL,\ \mu)$, which is equivalent to $M(L) = LM(1)$ in law.[7] Henceforth, we let $L = 1$ without any loss of generality and thereby give simpler formulations of the SDSI equations, generalized star equation, and intermittency differentiation rule. Introducing the operator

$$\mathcal{L} = \frac{\mu}{2} z \frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial z}, \tag{46}$$

we can then simplify Eqs. (25), (26), (45), and (29) as follows

$$\mathcal{L} v(z,\ \mu) = \mathbf{E}\bigg[\exp\bigg(-z \int_0^1 s^{-\mu}\, dM(s)\bigg)\bigg], \tag{47}$$

$$\bigg(\mathcal{L} + \mu \frac{\partial}{\partial z}\bigg)\mathcal{L} v(z,\ \mu) = \mathbf{E}\bigg[\exp\bigg(-z \int_0^1 s^{-\mu}(1-s)^{-\mu}\, dM(s)\bigg)\bigg], \tag{48}$$

$$\frac{\partial}{\partial z} v(z,\ \mu) = -\int_0^1 \mathbf{E}\big[e^{-zM^{(u)}(1)}\big]\, du, \tag{49}$$

$$\frac{\partial}{\partial \mu} v(z,\ \mu) = -\frac{z^2}{2} \int_0^1 \int_0^1 \mathbf{E}\bigg[\exp\bigg(-z \int_0^1 |s-s_1|^{-\mu} |s-s_2|^{-\mu}\, dM(s)\bigg)\bigg] \times |s_1 - s_2|^{-\mu} \log|s_1 - s_2|\, d\vec{\mathbf{s}}^{(2)}. \tag{50}$$

The SDSI equations, Eqs. (47) and (48), are simpler than the generalized star equation, Eq. (49), as they only involve one path integral, whereas Eq. (49) involves one per $u \in [0,\ 1]$, *i.e.* infinitely many. The obvious difficulty that they all share is that they are not single-variable equations as they involve the *entire path* of $M(s)$ for $s \in [0,\ 1]$. It remains an open problem to eliminate this path dependence so as to derive a single-variable equations for the Laplace transform.

We end this section with a remark concerning the Selberg integral, confer Eq. (39) above. Following Ref. 1, we introduce the notation

$$S_n(\alpha, \beta, \gamma) = \int_0^1 \cdots \int_0^1 \prod_{k=1}^n s_k^{\alpha-1}(1-s_k)^{\beta-1} \prod_{i<j}^n |s_i - s_j|^{2\gamma}\, d\vec{\mathbf{s}}^{(n)}. \tag{51}$$

[7] This is also a corollary of Eq. (14).

Then, $\mathbf{E}[M(1)^n] = S_n(1, 1, -\mu/2)$. It is not difficult to see that Eqs. (47) and (48) imply the following recurrence relations for the Selberg integral

$$\left(1 - \frac{\mu}{2}n\right)S_{n+1}\left(1, 1, -\frac{\mu}{2}\right) = S_n\left(1 - \mu, 1, -\frac{\mu}{2}\right), \tag{52}$$

$$\left(1 - \frac{\mu}{2}(n+2)\right)S_{n+1}\left(1 - \mu, 1, -\frac{\mu}{2}\right) = S_n\left(1 - \mu, 1 - \mu, -\frac{\mu}{2}\right). \tag{53}$$

Finally, Eq. (49) implies yet another recurrence relation

$$S_{n+1}(1, 1, -\mu/2) = \int_0^1 du\left[\int_0^1 \cdots \int_0^1 \prod_{k=1}^n |s_k - u|^{-\mu} \prod_{i<j}^n |s_i - s_j|^{-\mu} d\vec{\mathbf{s}}^{(n)}\right]. \tag{54}$$

All of these recurrences appear to be new.

7. THE LOW INTERMITTENCY LIMIT

The key expansions in Eqs. (37) and (40) are exact. They require the knowledge of the coefficients $\int_{\otimes k} h_{n,k}(\vec{\mathbf{s}})\, d\vec{\mathbf{s}}^{(k)}$, which can all be computed in principle by the iterative rule of Corollary 5.1. As such computations are not easy, we will restrict our attention here to the terms of order one: $h_{1,1}$ and $h_{1,2}$. Specifically, we will compute the Laplace transform of $M(1)$ and the following related functionals

$$\int_0^1 s^{-\mu}\, dM(s), \quad \int_0^1 s^{-\mu}(1-s)^{-\mu}\, dM(s), \quad M^{(u)}(1), \quad u \in [0,\ 1],$$

which enter Eqs. (47)–(49), up to the first nontrivial terms in the intermittency parameter μ.

Proposition 7.1.

$$\mathbf{E}[\exp(-zM(1))] = e^{-z}\left(1 + \frac{3}{4}\mu z^2\right) + o(\mu), \tag{55}$$

$$\mathbf{E}\left[\exp\left(-z\int_0^1 s^{-\mu}\, dM(s)\right)\right] = e^{-z}\left(1 - \mu z + \frac{3}{4}\mu z^2\right) + o(\mu), \tag{56}$$

$$\mathbf{E}\left[\exp\left(-z\int_0^1 s^{-\mu}(1-s)^{-\mu}\, dM(s)\right)\right] = e^{-z}\left(1 - 2\mu z + \frac{3}{4}\mu z^2\right) + o(\mu), \tag{57}$$

$$\mathbf{E}\left[e^{-zM^{(u)}(1)}\right] = e^{-z}\left(1 + \mu z[u\log u + (1-u)\log(1-u) - 1] + \frac{3}{4}\mu z^2\right) + o(\mu). \tag{58}$$

The expressions in Eqs. (55)–(58) obey Eqs. (47)–(49) up to $o(\mu)$.

The proof follows directly from Corollary 5.2.

We end this section with a similar result for the Selberg integral.

Proposition 7.2. *Given* $0 \leq \mu < 1$ *and* $2 \leq m < 2/\mu$,

$$\int_0^1 \cdots \int_0^1 \prod_{i<j}^{m} |s_i - s_j|^{-\mu} d\vec{\mathbf{s}}^{(m)} = 1 + \frac{3}{4}\mu\, m(m-1) + o(\mu). \tag{59}$$

This is a corollary of Proposition 5.2.

The higher order coefficients are analytically computable and appear to have a number theoretic content. Indeed, they can be expressed in terms of values of the Riemann zeta function on positive integers. The computation is quite lengthy and will not be given here.

8. THE PROOFS

In this section we will present detailed proofs of most of our results as well as several key lemmas. For the convenience of the reader, the proofs of propositions from Sec. 6 follow right after those from Sec. 4 as they are based on the same technique. The proofs of Propositions 5.1 and 5.2 are given at the end of the section.

Throughout this section we will use $g_{L,\varepsilon}(s_1, s_2)$ to denote the ε truncation of the $g(s_1, s_2)$ function defined in Eq. (23) so that $\mathbf{Cov}(\omega_\varepsilon(s_1), \omega_\varepsilon(s_2)) \triangleq \mu\, g_{L,\varepsilon}(s_1, s_2)$[8] and $\lim_{\varepsilon\to 0} g_{L,\varepsilon}(s_1, s_2) = g(s_1, s_2)$.

We begin with an auxiliary lemma that extends the classical theorem of Girsanov.[(12)]

Lemma 8.1. *Let* $\omega(s)$ *be a gaussian process defined on an interval* $s \in [0,\ L]$, *which has continuous sample paths and satisfies*

$$\mathbf{E}[\exp(\omega(s))] = 1 \tag{60}$$

for all s. *Let* s_1 *and* s_2 *be any two distinct times,* $s_1,\ s_2 \in [0,\ L]$, *and let* $C(s,\ t) \triangleq \mathbf{Cov}(\omega(s),\ \omega(t))$ *denote the covariance function of* $\omega(s)$, *which is assumed to be continuous. Let* $u(z,\ f)$ *denote the exponential functional*

$$u(z,\ f) \triangleq \exp\left(-z \int_0^L e^{f(s)+\omega(s)}\, ds\right), \tag{61}$$

[8] The L subscript is there to emphasize the dependence on L, which is important in the proof of Proposition 5.1 below.

where f is an arbitrary continuous function. Then,

$$\mathbf{E}[u(z,\ f + C(\cdot, s_1))] = \mathbf{E}[u(z,\ f)\,\mathrm{e}^{\omega(s_1)}], \tag{62}$$

$$\mathbf{E}[u(z,\ f + C(\cdot, s_1) + C(\cdot, s_2))] = \mathrm{e}^{-C(s_1,s_2)}\mathbf{E}\big[u(z,\ f)\,\mathrm{e}^{\omega(s_1)+\omega(s_2)}\big], \tag{63}$$

$$\int_0^L \mathrm{e}^{f(s)}\mathbf{E}[u(z,\ f + C(\cdot, s))]\,ds = -\frac{\partial}{\partial z}\mathbf{E}[u(z,\ f)]. \tag{64}$$

Proof: Introduce an equivalent probability measure

$$d\mathcal{Q} \stackrel{\triangle}{=} \mathrm{e}^{\omega(s_1)}\,d\mathcal{P},$$

where $\mathcal{P}$ is the original probability measure corresponding to $\mathbf{E}$. Then, the law of the process $s \to \omega(s) + C(s,\ s_1)$ with respect to $\mathcal{P}$ equals the law of the original process $s \to \omega(s)$ with respect to $\mathcal{Q}$. Indeed, it is easy to show that the two processes have the same finite-dimensional distributions by computing their characteristic functions. The computation, which is similar to the argument given in Ref. 27, is straightforward and will be omitted. The continuity of sample paths can then be used to conclude that the equality of all finite-dimensional distributions implies the equality in law. An alternative argument is given in the appendix of Ref. 28.

To verify Eq. (63), it is sufficient to remark that the measure

$$d\mathcal{Q} \stackrel{\triangle}{=} \mathrm{e}^{\omega(s_1)+\omega(s_2)-C(s_1,s_2)}\,d\mathcal{P}$$

is a probability measure that is equivalent to the original probability measure $\mathcal{P}$. It follows that the finite-dimensional distributions of the process $s \to \omega(s) + C(s,\ s_1) + C(s,\ s_2)$ with respect to $\mathcal{P}$ are the same as those of the original process $s \to \omega(s)$ with respect to $\mathcal{Q}$. The rest of the argument goes through verbatim.

Equation (64) follows directly from Eq. (62). □

Lemma 8.2. *The gaussian process $\omega_{\mu,L,\varepsilon}(s)$ defined in Eqs. (4–6) has continuous sample paths.*

Proof: As it is stationary gaussian, it is sufficient to show that its covariance function $r(\tau) \stackrel{\triangle}{=} C(0,\ \tau)$ is continuous and obeys the asymptotic

$$|r(\tau) - r(0)| = O(|\log|\tau||^{-q}),\ \ \tau \to 0,$$

for some $q > 3$, confer Theorem 9.2.1 in Ref. 8 and Theorem 3.5.7 in Ref. 11. In our case, the covariance function is manifestly continuous and $|r(\tau) - r(0)| = \mu|\tau|/\varepsilon$ for $|\tau| \le \varepsilon$ clearly satisfies this asymptotic. □

Proof of Proposition 4.1: The argument relies on a special invariance property of the $\omega_\varepsilon(s)$ process in Eqs. (4)–(6) combined with the backward Kolmogorov equation and Lemma 8.1. We will write $\omega_{L,\varepsilon}(s)$ to emphasize the dependence of the process' covariance on L. Let $f(s)$ be an arbitrary smooth function that does not involve μ and $v_\varepsilon(z, L, f)$ denote the generalized Laplace transform of the random measure $dM_\varepsilon(s)$ as in Sec. 4. Let

$$u_\varepsilon(z, L, f) \triangleq \exp\left(-z \int_0^L e^{\mu f(s)+\omega_{L,\varepsilon}(s)}\, ds \right) \tag{65}$$

so that $v_\varepsilon(z, L, f) = \mathbf{E}[u_\varepsilon(z, L, f)]$. Let us consider the limit

$$A \triangleq \frac{\partial}{\partial\delta}\bigg|_{\delta=0} \mathbf{E}^*\big[v_\varepsilon\big(ze^{B^*(\delta)}, L, f\big)\big], \tag{66}$$

where $B^*(t)$ is the standard Brownian motion independent of $\omega_{L,\varepsilon}(s)$ and $\mathbf{E}^*$ denotes the expectation with respect to $B^*(t)$. By the backward Kolmogorov equation, we have

$$A = \frac{1}{2}\frac{\partial^2}{\partial x^2}\bigg|_{x=0} v_\varepsilon\,(ze^x, L, f) = \frac{1}{2}\left[z\frac{\partial}{\partial z} + z^2\frac{\partial^2}{\partial z^2}\right] v_\varepsilon(z, L, f). \tag{67}$$

On the other hand, we have the following equality in law of stochastic processes viewed as random functions of s on the interval $[0, L]$ at fixed δ, μ, and ε

$$B^*(\delta) + \omega_{L,\varepsilon}(s) = \omega_{Le^{\delta/\mu},\varepsilon}(s) + \frac{\delta}{2}. \tag{68}$$

This is the decorrelation length invariance and is the first of four such invariances that are established in this paper. Since both processes are gaussian, it is sufficient to show that their means and covariances coincide on the interval $s \in [0, L]$, which follows by inspection from Eqs. (4)–(6). Now, we have from Eq. (68) the equality in law

$$\mathrm{e}^{B^*(\delta)} \int_0^L \mathrm{e}^{\mu f(s)+\omega_{L,\varepsilon}(s)} ds = \mathrm{e}^{\frac{\delta}{2}} \int_0^L \mathrm{e}^{\mu f(s)+\omega_{Le^{\delta/\mu},\varepsilon}(s)} ds. \tag{69}$$

By rewriting the integral in Eq. (69) as $\int_0^L \cdots ds = \int_0^{Le^{\delta/\mu}} \cdots ds - \int_L^{Le^{\delta/\mu}} \cdots ds$, it follows that the limit in Eq. (66) equals

$$\begin{aligned} A = {} & \frac{1}{2} z\frac{\partial v_\varepsilon}{\partial z} + \frac{L}{\mu}\frac{\partial v_\varepsilon}{\partial L} + \frac{zL}{\mu}\mathrm{e}^{\mu f(L)}\,\mathbf{E}\big[u_\varepsilon(z, L, f)e^{\omega_{L,\varepsilon}(L)}\big] \\ & + zL \int_0^L \frac{\partial f}{\partial L}(s)\,\mathrm{e}^{\mu f(s)}\,\mathbf{E}\big[u_\varepsilon(z, L, f)\mathrm{e}^{\omega_{L,\varepsilon}(s)}\big]\, ds. \end{aligned} \tag{70}$$

By Lemma 8.1, this simplifies to

$$A = \frac{1}{2}z\frac{\partial v_\varepsilon}{\partial z} + \frac{L}{\mu}\frac{\partial v_\varepsilon}{\partial L} + \frac{zL}{\mu}\mathrm{e}^{\mu f(L)}\, v_\varepsilon(z,\ L,\ f + g_{L,\varepsilon}(\cdot, L))$$
$$+ zL\int_0^L \frac{\partial f}{\partial L}(s)\,\mathrm{e}^{\mu f(s)}\, v_\varepsilon(z,\ L,\ f + g_{L,\varepsilon}(\cdot, s))\,ds. \tag{71}$$

To complete the proof, it remains to equate Eqs. (67) and (71) and take the $\varepsilon \to 0$ limit. □

Proof of Corollary 4.1: The result follows from Proposition 4.1 by setting $f \equiv 0$. □

Proof of Corollary 4.2: By Proposition 4.1 with $f(s) = g(s,\ 0)$ we obtain

$$\left(\frac{\mu}{2L}z^2\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial L}\right)v(z, L, g(\cdot, 0)) = zv(z, L, g(\cdot, 0) + g(\cdot, L))$$
$$+ \frac{\mu}{L}z\int_0^L e^{\mu g(u,0)} v(z, L, g(\cdot, 0) + g(\cdot, u))\,du. \tag{72}$$

The integral in Eq. (72) equals $-\partial/\partial z\ v(z,\ L,\ g(\cdot, 0))$ by Lemma 8.1, Eq. (64). By symmetry, $v(z,\ L,\ g(\cdot, L)) = v(z,\ L,\ g(\cdot, 0))$. The result now follows from Corollary 4.1 using the identity

$$\left(\frac{\mu}{2L}z^2\frac{\partial^2}{\partial z^2} + \frac{\mu}{L}z\frac{\partial}{\partial z} - \frac{\partial}{\partial L}\right)\frac{1}{z}\left(\frac{\mu}{2L}z^2\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial L}\right) = \frac{1}{z}\left(\frac{\mu}{2L}z^2\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial L}\right)^2. \tag{73}$$

□

Proof of Proposition 4.2: The argument follows exactly the same steps as in the proof of Proposition 4.1 and its corollaries. Instead of Eq. (68) we now have

$$B^*(\delta) + \omega_L(s) = \omega_{L+\frac{\delta}{\mu}}(s) + \frac{\delta}{2}, \tag{74}$$

where

$$\omega_L(s) = \sqrt{\mu}(B(L+s) - B(s)) - \mu\frac{L}{2} \tag{75}$$

is the process that appears in Eq. (19). The argument is completed by evaluating the A limit in the two ways. The remaining details are straightforward. □

Proof of the Classical Feynman-Kac formula, Eq. (21): The proof of Proposition 4.1 applies to the exponential functional of the standard Brownian motion. Instead of Eq. (68), the required invariance is simply $B^*(\delta) + B(s) = B(s + \delta)$. □

Proof of Proposition 6.1: The proof follows the same pattern as the proof of Proposition 4.1 but requires a different invariance. Instead of Eq. (68), we need the following truncation scale invariance

$$B^*(\delta) + \omega_{L,\varepsilon}(s) = \omega_{L,\varepsilon e^{-\delta/\mu}}(se^{-\delta/\mu}) + \frac{\delta}{2}. \tag{76}$$

As before, this is verified by comparing the means and variances of the two processes. The argument is completed by evaluating the A limit in the two ways. Details are straightforward and will me omitted. □

Proof of Proposition 6.2: Again, the proof is based on evaluating the A limit in the two ways. We need yet another invariance principle

$$B^*(\delta) + \omega_{L,\varepsilon}(s) = \omega_{Le^{2\delta/\mu},\varepsilon e^{\delta/\mu}}(se^{\delta/\mu}) + \frac{\delta}{2}, \tag{77}$$

which is verified as usual. Proceeding as before, we obtain the equation

$$\left(t\frac{\partial}{\partial t} - \frac{\mu}{2}z^2\frac{\partial^2}{\partial z^2} - z\frac{\partial}{\partial z} + 2L\frac{\partial}{\partial L}\right)v(z,\ \mu,\ t,\ L) = 0. \tag{78}$$

Combining this with Eq. (42), we arrive at Eq. (43). Equation (44) is an obvious corollary of Eq. (43) due to $v(z,\ \mu,\ L) = v(z,\ \mu,\ L,\ L)$. □

Proof of Proposition 6.3: This is a special case of Lemma 8.1, Eq. (64). □

We now proceed to the proof of Proposition 5.1. The general idea of proof is similar to that of Proposition 4.1 but the details are much more difficult and require two additional computations that are explained in Lemmas 8.3 and 8.4 below. Of the previous results, we only require Lemmas 8.1 and 8.2. We remind the reader that $\mathbf{Cov}(\omega_\varepsilon(s_1),\ \omega_\varepsilon(s_2)) \stackrel{\triangle}{=} \mu\, g_{L,\varepsilon}(s_1, s_2)$.

Lemma 8.3. *Let $\omega_{\mu,L,\varepsilon}(s)$ be the gaussian process of Eqs. (4)–(6) and $f(\mu, s)$ be an arbitrary continuous function that vanishes as $\mu \to 0$. Let*

$$\mathcal{B}(s) \stackrel{\triangle}{=} e^{f(\mu,s)+\omega_\varepsilon(s)} - 1. \tag{79}$$

Then, given any distinct $s_1, \ldots, s_k \in [0,\ L]$,

$$\mathbf{E}[\mathcal{B}(s_1)\mathcal{B}(s_2)] = \left(e^{f(\mu,s_1)} - 1\right)\left(e^{f(\mu,s_2)} - 1\right) + \mu\, g_{L,\varepsilon}(s_1, s_2) + o(\mu), \tag{80}$$

$$\mathbf{E}[\mathcal{B}(s_1)\cdots\mathcal{B}(s_k)] = \left(e^{f(\mu,s_1)} - 1\right)\cdots\left(e^{f(\mu,s_k)} - 1\right) + o(\mu), \quad k \neq 2, \tag{81}$$

as $\mu \to 0$.

Proof: It is easy to show from Eqs. (4)–(6) that for any subset σ of $\{1, \ldots, k\}$

$$\mathbf{E}\left[\exp\left(\sum_{i\in\sigma}\omega_\varepsilon(s_i)\right)\right] = \exp\left(\sum_{\substack{i<j,\\ i,j\in\sigma}}\mu\, g_{L,\varepsilon}(s_i, s_j)\right). \tag{82}$$

Let $\mathcal{S}_p$ denote the set of all subsets of $\{1, \ldots, k\}$ that consist of exactly p indices, $p = 0 \cdots k$, with the convention that the only element of S_0 is the empty set. Then, given k distinct numbers, we have the algebraic identity

$$(a_1 - 1)\cdots(a_k - 1) = \sum_{p=0}^{k}(-1)^{k-p}\sum_{\sigma\in\mathcal{S}_p}\prod_{i\in\sigma}a_i, \tag{83}$$

taking all empty sums to mean zero and empty products to mean one. It is easily verified by induction. If we now expand the brackets on the left-hand side of Eq. (81) and make use of Eqs. (82) and (83), we obtain

$$\sum_{p=0}^{k}(-1)^{k-p}\sum_{\sigma\in\mathcal{S}_p}\exp\left(\sum_{i\in\sigma}f(\mu, s_i) + \mu\sum_{\substack{i<j,\\ i,j\in\sigma}}g_{L,\varepsilon}(s_i, s_j)\right). \tag{84}$$

It remains to expand this expression in μ and recall that $f(\mu, s) \to 0$ as $\mu \to 0$ by assumption. There results

$$\sum_{p=0}^{k}(-1)^{k-p}\sum_{\sigma\in\mathcal{S}_p}\mathrm{e}^{\sum_{i\in\sigma}f(\mu,s_i)} + \mu\sum_{p=0}^{k}(-1)^{k-p}\sum_{\sigma\in\mathcal{S}_p}\sum_{\substack{i<j,\\ i,j\in\sigma}}g_{L,\varepsilon}(s_i, s_j) + o(\mu). \tag{85}$$

By Eq. (83), the first term in Eq. (85) is exactly $\prod_{i=1}^{k}(\exp(f(\mu, s_i)) - 1)$ that occurs on the right-hand side of Eq. (81). It is not difficult to see that the second term in Eq. (85) equals $\mu\, g_{L,\varepsilon}(s_1, s_2)$ if $k = 2$ and is zero otherwise. □

Lemma 8.4. *Let $\omega_{\mu,L,\varepsilon}(s)$ be the gaussian process of Eqs. (4)–(6) and $f(s)$ be an arbitrary continuous function that does not involve μ. Let*

$$u_\varepsilon(z,\ \mu,\ f)\ \exp\left(-\, z\int_0^L \mathrm{e}^{\mu f(s)+\omega_\varepsilon(s)}\, ds\right). \tag{86}$$

Then, there holds the following identity

$$\frac{\partial}{\partial\mu}u_\varepsilon(z,\ \mu,\ f) = -zu_\varepsilon(z,\ \mu,\ f)\int_0^L \mathrm{e}^{\mu f(s)+\omega_\varepsilon(s)} f(s)\, ds$$

$$-\lim_{\delta\to 0}\left[\frac{u_\varepsilon(z,\ \mu,\ f)}{\delta}\sum_{k=1}^{\infty}\frac{(-z)^k}{k!}\left(\int_0^L e^{\mu f(s)+\omega_\varepsilon(s)}\left(e^{\mathcal{A}_\varepsilon(s)} - 1\right) ds\right)^k\right], \tag{87}$$

where

$$\mathcal{A}_\varepsilon(s) \overset{\triangle}{=} \omega_{\mu-\delta,L,\varepsilon}(s) - \omega_{\mu,L,\varepsilon}(s). \tag{88}$$

Proof: The result follows from representing $u_\varepsilon(z,\ \mu-\delta,\ f)$ as

$$\delta z u_\varepsilon(z,\ \mu,\ f)\int_0^L \mathrm{e}^{\mu f(s)+\omega_\varepsilon(s)}\,ds + \mathrm{e}^{-z\int_0^L \mathrm{e}^{\mu f(s)+\omega_{\mu,L,\varepsilon}(s)}(1+\mathrm{e}^{\mathcal{A}_\varepsilon(s)}-1)\,ds} + o(\delta)$$

and expanding the second term in powers of the "small" parameter

$$\int_0^L e^{\mu f(s)+\omega_\varepsilon(s)}\left(\mathrm{e}^{\mathcal{A}_\varepsilon(s)}-1\right)ds$$

that vanishes as $\delta \to 0$. □

Proof of Proposition 5.1: Let $v_\varepsilon(z,\mu,\ f)$ denote the generalized Laplace transform of $dM_\varepsilon(s)$ measure so that $v(z,\ \mu,\ f) = \lim_{\varepsilon\to 0} v_\varepsilon(z,\ \mu,\ f)$.[9] The starting point is the limit

$$A \overset{\triangle}{=} \frac{\partial}{\partial\delta}\bigg|_{\delta=0} \mathbf{E}^*\left[v_\varepsilon\left(ze^{B^*(\delta)},\ \mu,\ f\right)\right], \tag{89}$$

where $B^*(t)$ is the standard Brownian motion independent of $\omega_\varepsilon(s)$. By the backward Kolmogorov equation, we have

$$A = \frac{1}{2}\frac{\partial^2}{\partial x^2}\bigg|_{x=0} v_\varepsilon\left(ze^x,\ \mu,\ f\right) = \frac{1}{2}\left[z\frac{\partial}{\partial z} + z^2\frac{\partial^2}{\partial z^2}\right]v_\varepsilon(z,\ \mu,\ f). \tag{90}$$

On the other hand, this limit can be computed in a different way. There holds the following equality in law of stochastic processes viewed as random functions of s on the interval $[0,\ L]$ at fixed $0 < \delta < \mu$ and ε

$$B^*(\delta) + \omega_{\mu,L,\varepsilon}(s) = \omega_{\mu-\delta,L,\varepsilon}(s) + \bar{\omega}_{\delta,eL,\varepsilon}(s) + \frac{\delta}{2}, \tag{91}$$

where $\bar{\omega}_\varepsilon(s)$ denotes an independent copy of the $\omega_\varepsilon(s)$ process at the intermittency parameter δ and rescaled decorrelation length eL. This is the intermittency parameter invariance. As both processes are gaussian, it is sufficient to show that their means and covariance functions coincide, which follows by inspection from Eqs. (4)–(6). There results the identity in law

$$\mathrm{e}^{B^*(\delta)}\int_0^L \mathrm{e}^{\mu f(s)+\omega_{\mu,L,\varepsilon}ds} = \mathrm{e}^{\frac{\delta}{2}}\int_0^L \mathrm{e}^{\mu f(s)+\omega_{\mu-\delta,L,\varepsilon}(s)+\bar{\omega}_{\delta,eL,\varepsilon}(s)}ds.$$

[9] As we did in Sec. 5, we leave out L and include μ in the list of arguments.

Thus, to compute the limit in Eq. (89) we need to expand

$$\exp\Bigg(-z e^{\frac{\delta}{2}} \int_0^L e^{\mu f(s)+\omega_{\mu-\delta,L,\varepsilon}(s)+\bar{\omega}_{\delta,eL,\varepsilon}(s)} ds\Bigg) \tag{92}$$

in δ up to $o(\delta)$ terms. Let $\mathcal{A}_\varepsilon(s)$ be as in Eq. (88) and

$$\bar{\mathcal{A}}_\varepsilon(s) \triangleq \bar{\omega}_{\delta,eL,\varepsilon}(s). \tag{93}$$

While we do not know how to expand either $\mathcal{A}_\varepsilon(s)$ or $\bar{\mathcal{A}}_\varepsilon(s)$ in δ, they both clearly vanish as $\delta \to 0$. It follows that the expression in Eq. (92) can be written as

$$e^{-z\int_0^L e^{\mu f(s)+\omega_{\mu,L,\varepsilon}(s)}ds}\Big[1 - \delta\frac{z}{2}\int_0^L e^{\mu f(s)+\omega_{\mu,L,\varepsilon}(s)}ds + \sum_{k=1}^{\infty}\frac{(-z)^k}{k!}\mathcal{C}^k\Big], \tag{94}$$

$$\mathcal{C} \triangleq \int_0^L e^{\mu f(s)+\omega_{\mu,L,\varepsilon}(s)}\big(e^{\mathcal{A}_\varepsilon(s)+\bar{\mathcal{A}}_\varepsilon(s)} - 1\big)\, ds, \tag{95}$$

up to $o(\delta)$ terms. The advantage of this representation is that the only $\bar{\omega}_\varepsilon$ dependence is in $\bar{\mathcal{A}}_\varepsilon(s)$. This allows us to compute the $\mathbf{E}^*$ expectation in Eq. (89). Indeed, Eq. (89) entails two expectations: the $\mathbf{E}$ with respect to ω_ε process inherited from the definition of $v_\varepsilon(z, \mu, f)$ and the $\mathbf{E}^*$ expectation with respect to $\bar{\omega}_\varepsilon$ process. Interchanging their order, it follows from Eq. (94) that computing the $\mathbf{E}^*$ expectation is now reduced to computing $\mathbf{E}^*[\mathcal{C}^{\mathbf{k}}]$. As $\mathcal{A}_\varepsilon(s)$ and $\bar{\mathcal{A}}_\varepsilon(s)$ are independent processes, it follows from Lemma 8.3 applied to $\mathcal{B}(s) = \exp(\mathcal{A}_\varepsilon(s) + \bar{\mathcal{A}}_\varepsilon(s)) - 1$ that the $\mathbf{E}^*$ expectation equals

$$\mathbf{E}^*[\mathcal{B}(s_1)\mathcal{B}(s_2)] = \big(e^{\mathcal{A}_\varepsilon(s_1)} - 1\big)\big(e^{\mathcal{A}_\varepsilon(s_2)} - 1\big) + \delta\, g_{eL,\varepsilon}(s_1, s_2) + o(\delta), \tag{96}$$

$$\mathbf{E}^*[\mathcal{B}(s_1)\cdots\mathcal{B}(s_k)] = \big(e^{\mathcal{A}_\varepsilon(s_1)} - 1\big)\cdots\big(e^{\mathcal{A}_\varepsilon(s_k)} - 1\big) + o(\delta), \qquad k \neq 2. \tag{97}$$

We have now everything we need to compute the limit in Eq. (89). To simplify the following formulas, we will use $u_\varepsilon(z, \mu, f)$ introduced in Eq. (86) so that $v_\varepsilon(z, \mu, f) = \mathbf{E}[u_\varepsilon(z, \mu, f)]$ and write $\otimes k$ to denote the multiple integral $\int_0^L \cdots \int_0^L d\vec{\mathbf{s}}^{(k)}$. Collecting what we have shown so far, we obtain

$$\mathbf{E}^*\left[v_\varepsilon\left(z e^{B^*(\delta)}, \mu, f\right)\right] - v_\varepsilon(z, \mu, f) = \delta\frac{z}{2}\frac{\partial}{\partial z}v_\varepsilon(z, \mu, f) \tag{98}$$

$$+ \sum_{k=1}^{\infty}\frac{(-z)^k}{k!}\mathbf{E}\left[u_\varepsilon(z, \mu, f)\left(\int_{\otimes 1} e^{\mu f(s)+\omega_\varepsilon(s)}\left(e^{\mathcal{A}_\varepsilon(s)} - 1\right) ds\right)^k\right] \tag{99}$$

$$+ \delta\frac{z^2}{2}\mathbf{E}\left[u_\varepsilon(z, \mu, f)\int_{\otimes 2} e^{\mu f(s_1)+\omega_\varepsilon(s_1)+\mu f(s_2)+\omega_\varepsilon(s_2)}\, g_{eL,\varepsilon}(s_1, s_2) d\vec{\mathbf{s}}^{(2)}\right] + o(\delta). \tag{100}$$

It remains to divide by δ and take the limit $\delta \to 0$. By Lemma 8.4, we have

$$\lim_{\delta\to 0}\frac{1}{\delta}\sum_{k=1}^{\infty}\frac{(-z)^k}{k!}\mathbf{E}\left[u_\varepsilon(z,\ \mu,\ f)\left(\int_{\otimes 1} e^{\mu f(s)+\omega_\varepsilon(s)}\left(e^{\mathcal{A}_\varepsilon(s)}-1\right)ds\right)^k\right], \tag{101}$$

$$= -\frac{\partial}{\partial\mu}v_\varepsilon(z,\ \mu,\ f) - z\int_{\otimes 1} e^{\mu f(s)} f(s)\mathbf{E}[u_\varepsilon(z,\ \mu,\ f)e^{\omega_\varepsilon(s)}]\,ds, \tag{102}$$

$$= -\frac{\partial}{\partial\mu}v_\varepsilon(z,\ \mu,\ f) - z\int_{\otimes 1} e^{\mu f(s)} f(s)v_\varepsilon(z,\ \mu,\ f+g_{L,\varepsilon}(\cdot,s))\,ds, \tag{103}$$

where the last equality follows from Lemma 8.1. As for the expression in Eq. (100), we can write $g_{eL,\varepsilon}(s_1,s_2) = 1 + g_{L,\varepsilon}(s_1,\ s_2)$ resulting in

$$\frac{z^2}{2}\frac{\partial^2}{\partial z^2}v_\varepsilon(z,\ \mu,\ f) + \frac{z^2}{2}\int_{\otimes 2} e^{\mu(f(s_1)+f(s_2)+g_{L,\varepsilon}(s_1,s_2))}g_{L,\varepsilon}(s_1,\ s_2)$$
$$\times\, v_\varepsilon(z,\ \mu,\ f+g_{L,\varepsilon}(\cdot,s_1)+g_{L,\varepsilon}(\cdot,s_2))\,d\vec{\mathbf{s}}^{(2)} \tag{104}$$

by Lemma 8.1. Collecting all terms, we finally obtain for the limit in Eq. (89)

$$\frac{z}{2}\frac{\partial}{\partial z}v_\varepsilon(z,\mu,f) + \frac{z^2}{2}\frac{\partial^2}{\partial z^2}v_\varepsilon(z,\mu,f) - z\int_{\otimes 1} e^{\mu f(s)} f(s)v_\varepsilon(z,\mu,f+g_{L,\varepsilon}(\cdot,s))\,ds$$
$$-\frac{\partial}{\partial\mu}v_\varepsilon(z,\ \mu,\ f) + \frac{z^2}{2}\int_{\otimes 2} e^{\mu(f(s_1)+f(s_2)+g_{L,\varepsilon}(s_1,s_2))}g_{L,\varepsilon}(s_1,\ s_2)$$
$$\times\, v_\varepsilon(z,\ \mu,\ f+g_{L,\varepsilon}(\cdot,s_1)+g_{L,\varepsilon}(\cdot,s_2))\,d\vec{\mathbf{s}}^{(2)}. \tag{105}$$

The result follows by comparing this formula with the expression in Eq. (90) and taking the limit $\varepsilon \to 0$. □

Proof of Proposition 5.2: The starting point is the identity

$$\int_0^1\cdots\int_0^1\prod_{i<j}^{m}|s_i-s_j|^{-\mu}d\vec{\mathbf{s}}^{(m)} = (-1)^m\frac{\partial^m}{\partial z^m}\bigg|_{z=0}\mathbf{E}\left[e^{-zM(1)}\right] \tag{106}$$

that follows from Eq. (39) by letting $L = 1$. Now, the Leibnitz formula shows that

$$\frac{\partial^m}{\partial z^m}\bigg|_{z=0}[e^{-z}z^k] = (-1)^{m-k}\frac{m!}{(m-k)!},\quad k = 0\cdots m, \tag{107}$$

and is zero otherwise. The result now follows from Eq. (38). □

9. CONCLUSIONS

We presented a new approach to the distribution of the limit lognormal process. The approach is based on novel functional Feynman-Kac equations for the Laplace transform of the limit distribution. The equations capture the stochastic dependence structure of increments (SDSI) of the process and quantify how its distribution varies with the intermittency parameter.

The origin of the Feynman-Kac equations is rooted in special invariance identities of the stationary gaussian process, which underlies the limit lognormal construction, combined with the backward Kolmogorov equation and a generalized Girsanov theorem. We established four such identities that capture invariances of this gaussian process with respect to the decorrelation length, intermittency parameter, and truncation scale.

Our main result is twofold. First, we showed that the SDSI of the limit lognormal process has two layers of complexity, each of which can be described by a functional Feynman-Kac equation that follows from the decorrelation length invariance. As we illustrated in the case of the Schmitt process, $M(L)$ can be seen as the limit of a sum of products of non-identically distributed, independent, recurrent, lognormal random weights. By removing the non-identically distributed condition, we introduced a toy limit lognormal process that has the same primary layer of complexity, *i.e.* independent, recurrent, and lognormal weights, as do the limit lognormal and Schmitt processes. The toy process is an interesting nonmarkovian stochastic process that shares with the original constructions the properties of having stationary increments and a finite decorrelation length.

The first pair of Feynman-Kac equations quantifies the primary layer of complexity. The mathematical structures of the equations for the limit lognormal and toy processes are essentially the same thereby reflecting the fact that the corresponding SDSIs share the primary complexity layer. The second pair of Feynman-Kac equations captures the secondary layer of complexity, and the mathematical structures that these equations exhibit are quite different. In the toy case, the equation is a single-variable equation, *i.e.* it involves the Laplace transform of the unknown distribution only. In the exact limit lognormal case, the equation does not reduce to a single-variable equation but rather contains terms involving the entire path of the process. The origin of the difference is the second layer of complexity that the toy process does not possess, *i.e.* the fact that the independent lognormal weights are not identically distributed in the exact construction. The phenomenon of path dependent terms is a reflection of the non-markovian structure of the underlying gaussian processes that define both the exact and toy limit lognormal processes. The underlying process in the limit lognormal case is very strongly non-markovian, while this process is only mildly non-markovian in the toy case making it possible to establish a single-variable equation for it and, in fact, compute its Laplace transform in a closed form.

Our second result is the exact computation of the generalized Laplace transform and integral moments of the limit lognormal process as power series expansions in the intermittency parameter. The coefficients of these expansions are computed iteratively by multiple integration *via* explicit three-term recursions. The origin of these expansions is the intermittency differentiation equation, which is nothing but a functional Feynman-Kac equation that corresponds to the intermittency invariance. It relates the derivative of the generalized Laplace transform with respect to the intermittency parameter to special functional shifts. While it is also not a single-variable equation, the intermittency differentiation equation differs from the SDSI equations in a very significant way: it does not involve derivatives with respect to the argument of the Laplace transform and so can be iterated infinitely many times. It is precisely this property that allows us to represent the generalized Laplace transform and moments as infinite series expansions. In particular, we obtained such expansions for the Laplace transform itself and all the path dependent terms that enter the SDSI equations, thereby effectively solving them exactly.

As side results, the method of functional Feynman-Kac equations allowed us to give simpler proofs for the main results that appeared first in Ref. 28 and to establish novel recurrence relations for the Selberg integral, which is known to represent the positive integral moments of the limit lognormal process.

Our results pose several interesting problems and open a new avenue for future research. First, we do not have a proof of convergence of our infinite series expansions. Such a proof requires computing all the coefficients $\int_{\otimes k} h_{n,k}(\vec{\mathbf{s}})\, d\vec{\mathbf{s}}^{(k)}$ or at least their asymptotics. In this paper we restricted ourselves to the order one coefficients only. The investigation of higher order coefficients is left to future studies, suffice it to say that they appear to have a number theoretic content. Second, another aspect of computing the coefficients is to understand how $h_{n,k}(\vec{\mathbf{s}})$ depend on the shifts $f(s)$. This knowledge would allow us to derive the SDSI equations from the intermittency differentiation equation. In fact, we computed the first nontrivial coefficients and verified that both the SDSI equations and as well as the generalized star equation hold in this approximation. It would be interesting to show that they continue to hold to all orders of the intermittency parameter, which would verify the claim that the intermittency differentiation equation contains in itself all the other equations. Finally, it still remains an open problem to derive a single-variable equation for the Laplace transform. This task is not so much important for computing the distribution but rather for gaining an insight into multifractal physics.

APPENDIX

In this section we will derive the probability density function (pdf) and Laplace transform of $M^{(\mathrm{toy})}(L)$. The derivation uses the "closeness" of the omega process in Eqs. (19) and (75) to being markovian that we already mentioned at the end of Sec. 4.

Proposition A.1. *Let $W(s)$ denote a standard Brownian motion starting at zero and let $\mathcal{N}$ denote an independent gaussian random variable with mean $-\mu\frac{L}{2}$ and variance $\mu\frac{L}{2}$. Then, the law of $M^{(\text{toy})}(L)$ is the same as that of*

$$\frac{2}{\mu}\,\mathrm{e}^{\mathcal{N}}\int_0^{\mu\frac{L}{2}}\mathrm{e}^{W\left(\mu\frac{L}{2}\right)-2W(s)}ds. \tag{A1}$$

Proof: We have the following equality in law of stochastic processes on the interval $s\in[0,\ L]$

$$\omega_L(s)=\mathcal{N}+W\left(\mu\frac{L}{2}\right)-2W\left(\mu\frac{s}{2}\right). \tag{A2}$$

The proof follows upon comparing the means and covariances of both processes. The result now follows by a change of variables. □

The task of computing the law of $M^{(\text{toy})}(L)$ is thus reduced to that of computing

$$\mathcal{Z}=\int_0^t\mathrm{e}^{W(t)-2W(s)}ds \tag{A3}$$

at $t=\mu L/2$, whose law is well-known. Indeed, exponential functionals of Brownian motion appear in one-dimensional disordered systems, confer Ref. 25 as well as in mathematical finance, confer Ref. 10. In particular, the pdf of $\mathcal{Z}$ is shown in Ref. 23 to be

$$\mathrm{pdf}^{(\mathcal{Z})}(z)=2K_0\left(\frac{1}{z}\right)\theta\left(\frac{1}{z},\ t\right)\frac{dz}{z}, \tag{A4}$$

where $K_0(r)$ denotes the modified Bessel function of the second kind of order zero and the function $\theta(r,\ t)$ is defined by

$$\theta(r,\ t)=\frac{r}{\sqrt{2\pi^3t}}\int_0^\infty\mathrm{e}^{(\pi^2-x^2)/2t}\mathrm{e}^{-r\cosh(x)}\sinh(x)\sin\left(\frac{\pi x}{t}\right)dx. \tag{A5}$$

As a corollary of Proposition A.1 we then obtain the pdf and Laplace transform of $M^{(\text{toy})}(L)$ in terms of those of $\mathcal{Z}$. Letting $t=\mu L/2$, we have

$$\mathrm{pdf}_{M^{(\text{toy})}(L)}(z)=\frac{\mu}{2}\int_{-\infty}^\infty\mathrm{pdf}^{(\mathcal{Z})}(z\mu\,\mathrm{e}^{t-y}/2)\frac{e^{-\frac{y^2}{2t}}}{\sqrt{2\pi t}}e^{t-y}\,dy, \tag{A6}$$

$$v^{(\text{toy})}(z,\ L)=\int_{-\infty}^\infty v^{(\mathcal{Z})}(2z\mathrm{e}^{y-t}/\mu)\frac{\mathrm{e}^{-\frac{y^2}{2t}}}{\sqrt{2\pi t}}\,dy. \tag{A7}$$

ACKNOWLEDGMENTS

The author would like to express gratitude to Professor F. Schmitt for bringing Ref. 32 to his attention. The author is also grateful to Professor S. Orszag for helpful discussions.

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